Problem 4

Find all functions f that satisfy the equation

$$\left(\int f(x)\,dx\right)\left(\int \frac{1}{f(x)}\,dx\right) = -1$$

Solution

Because there are two integrals in this equation, we will have to differentiate both sides of the equation twice to remove them. Recall that the fundamental theorem of calculus says that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Start by differentiating both sides of the equation. We have to use the product rule on the left since it is a product of two functions of x.

$$\frac{d}{dx}\left[\left(\int f(x)\,dx\right)\left(\int \frac{1}{f(x)}\,dx\right)\right] = \frac{d}{dx}(-1)$$

$$f(x)\left(\int \frac{1}{f(x)}\,dx\right) + \left(\int f(x)\,dx\right)\frac{1}{f(x)} = 0$$

$$f(x)\left(\int \frac{1}{f(x)}\,dx\right) = -\frac{1}{f(x)}\left(\int f(x)\,dx\right)$$

$$[f(x)]^2\left(\int \frac{1}{f(x)}\,dx\right) = (-1)\left(\int f(x)\,dx\right)$$

Make use of the starting equation to replace -1.

$$[f(x)]^2\cancel{\left(\int \frac{1}{f(x)}\,dx\right)} = \left(\int f(x)\,dx\right)\cancel{\left(\int \frac{1}{f(x)}\,dx\right)}\left(\int f(x)\,dx\right)$$

$$[f(x)]^2 = \left(\int f(x)\,dx\right)^2$$

$$f(x) = \pm\int f(x)\,dx$$

Now differentiate both sides for the second time.

$$\frac{df}{dx} = \pm f(x)$$

Separate variables.

$$\frac{df}{f} = \pm dx$$

$$\ln|f| = \pm x + C$$

$$|f| = e^{\pm x + C}$$

Therefore,

$$f(x) = Ae^{\pm x}.$$